MATHEMATICS

CHARACTERIZATION OF ANOSOV DIFFEOMORPHISMS

BY

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Let $f: M \to M$ be a C^1 diffeomorphism of a compact manifold. Let $Tf: TM \to TM$ denote the induced tangent mapping. One says (see [2]) that f is Anosov if and only if there exists a continuous direct sum splitting $TM = E_+ \oplus E_-$ such that E_+ and E_- are preserved by Tf and such that for some (and therefore every) Riemannian metric $\| \ \|$ on M, there exist $C>1$ and $0<\lambda<1$ such that $\|Tf^k(v)\| \leqq C\lambda^k \|v\|$ for all positive integers k and all $v \in E_-$ and $\|Tf^k(v)\| \geqq C^{-1}\lambda^{-k}\|v\|$ for all positive integers k and all $v \in E_+$.

For any vector bundle E over M, we let $\Gamma(E)$ denote the $\mathbf{R}$ vector space of continuous sections of E, where $\mathbf{R}$ denotes the field of real numbers. Let $f_* : \Gamma(TM) \to \Gamma(TM)$ be given by $f_*(\zeta) = Tf \circ \zeta \circ f^{-1}$.

In [2], we showed that if $f_* - I$ is an automorphism of $\Gamma(TM)$, then f is structurally stable. This is easily seen to imply Anosov's theorem that an Anosov diffeomorphism is structurally stable. The main result of this paper ($a \Rightarrow d$ in the theorem below) is that if $f_* - I$ is an automorphism, then f is Anosov, which shows that the result we obtained in [2] is not a true generalization of Anosov's theorem. (This fact was pointed out in [2].)

We set $X = \Gamma(TM) \otimes \hat{\mathbf{R}}\mathbf{C}$ and $L = f_* \otimes \mathbf{C} : X \to X$. Then X is a $\mathbf{C}$ vector space and L is $\mathbf{C}$ linear, where $\mathbf{C}$ denotes the field of complex numbers. Recall that the spectrum of L, $\sigma(L)$, is defined as the set of all $\lambda \in C$ such that $L - \lambda I : X \to X$ is not an automorphism.

Theorem. The following are equivalent.

a) $f_* - I$ is an automorphism.

b) $1 \notin \sigma(L)$.

c) $\lambda \notin \sigma(L)$ if $|\lambda| = 1$.

d) f is an Anosov diffeomorphism.

e) There exists a direct sum splitting $TM = E_- \oplus E_+$ such that E_+ and E_- are preserved by Tf, a Riemannian metric $\| \ \|$ on M, and λ satisfying $0<\lambda<1$ such that $\|Tf(v)\| \leqq \lambda\|v\|$ for all $v \in E_-$ and $\|Tf(v)\| \geqq \lambda^{-1}\|v\|$ for all $v \in E_+$.

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The proof goes $a \Rightarrow b \Rightarrow c \Rightarrow d \Rightarrow e \Rightarrow a$. Trivially $a \Leftrightarrow b$. Also $d \Rightarrow e$ is easily shown, as follows. Let $\| \ \|$ be a Riemannian metric on M. By hypothesis, there exists a direct sum splitting $TM = E_+ \oplus E_-$, preserved by Tf, and $C > 1$ and λ satisfying $0 < \lambda < 1$ such that $\|Tf^k(v)\| \leqq C\lambda^k \|v\|$ for all $v \in E_-$ and $\|Tf^k(v)\| \geqq C^{-1}\lambda^{-k}\|v\|$ for all $v \in E_+$. Take any μ satisfying $0 < \lambda < \mu < 1$ and choose K such that $C(\lambda/\mu)^K < 1$.

Define a new Riemannian metric $\| \ \|_1$ by

$$\|v\|_1^2 = \sum_{k=0}^{K-1} \|\mu^{-k} Tf^k(v)\|^2, \text{ if } v \in E_-$$

$$\|v\|_1^2 = \sum_{k=0}^{K-1} \|\mu^{-k} Tf^{-k}(v)\|^2, \text{ if } v \in E_+,$$

and generally for $v = v_- + v_+$, $v_- \in E_-$, $v_+ \in E_+$, $\|v\|_1^2 = \|v_-\|_1^2 + \|v_+\|_1^2$. Let $v \in E_-$. Then

$$\|Tf(v)\|_1^2 = \mu^2 \sum_{k=1}^{K} \|\mu^{-k} Tf^k(v)\|^2 = \mu^2[\|v\|_1^2 - \|v\|^2 + \|\mu^{-K} Tf^K(v)\|^2] < \mu^2 \|v\|_1^2,$$

because

$$\|\mu^{-K} Tf^K(v)\| \leqq C(\lambda/\mu)^K \|v\| < \|v\|.$$

Hence

$$\|Tf(v)\|_1 < \mu \|v\|_1.$$

The same argument applied to f^{-1} shows that $\|Tf(v)\|_1 \geqq \mu^{-1}\|v\|_1$ for all $v \in E_+$.

From now on, we let $\| \ \|$ denote a fixed Riemannian metric on M. This induces a Hermition norm on $TM \otimes_{\mathbf{R}} \mathbf{C}$, given by $\|v + iw\|^2 = \|v\|^2 + \|w\|^2$, for v, $w \in TM$, and norms on $\Gamma(TM)$ and $\Gamma(TM \otimes_{\mathbf{R}} \mathbf{C})$ given by

$$\|\zeta\| = \sup \{\|\zeta(x)\| : x \in M\}.$$

There is a canonical identification of $\mathbf{C}$ vector spaces $X = \Gamma(TM \otimes_{\mathbf{R}} \mathbf{C})$, given by $\zeta_1 + i\zeta_2 = \zeta_1 + i\zeta_2$ for $\zeta_1, \zeta_2 \in \Gamma(TM)$. We use this identification to induce a norm $\| \ \|$ on X. Then $\Gamma(TM)$ is a Banach space over $\mathbf{R}$, X is a Banach space over $\mathbf{C}$, and $X = \Gamma(TM) \otimes_{\mathbf{R}} \mathbf{C}$ as *topological* vector spaces.

Now we show $e \Rightarrow a$. We suppose the Riemannian metric chosen so that e is satisfied. Let

$$f_- = f_* | \Gamma(E_-) : \Gamma(E_-) \to \Gamma(E_-)$$

and

$$f_+ = f_* | \Gamma(E_+) : \Gamma(E_+) \to \Gamma(E_+).$$

Then $\|f_-\| < 1$, so $-\sum_{k=0}^{\infty} f_-^k$ converges. It clearly equals $(f_- - I)^{-1}$. Also f_+ has an inverse and $\|f_+^{-1}\| < 1$, so $(f_+^{-1} - I)^{-1}$ exists. But

$$(f_+ - I)^{-1} = -f_+^{-1}(f_+^{-1} - I)^{-1}$$

also exists. Since both $(f_- - I)^{-1}$ and $(f_+ - I)^{-1}$ exist, $(f_* - I)^{-1}$ exists.

In order to show $b \Rightarrow c$, we need to show that the non periodic points are dense in M. Let P_n denote the set of points $x \in M$ such that $f^n(x)=x$. By the Baire category theorems, it is enough to show that P_n has no interior points. (Since P_n is closed $\bigcup\limits_n P_n$ is of the first category, if P_n has no interior points). Suppose the contrary and let n be the smallest integer such that P_n has interior points. Let x be an interior point and have period n (so that $f^k(x)\neq x$ for $1\leqq k<n$). Then there exists a neighborhood U of x in P_n such that $U \cap f^k(U)=\emptyset$ for $0<k<n$. Let $\zeta_0 \in \Gamma(TM)$ have support in U and satisfy $\zeta_0(x)\neq 0$.

Let $\zeta = \sum\limits_{k=0}^{n-1} f_*^k(\zeta_0)$. Then $\zeta(x)=\zeta_0(x)$ (since $f_*^k(\zeta_0)$ has support in $f^{-k}(U)$ and $x \notin f^{-k}(U)$ for $0<k<n$), so ζ does not vanish. Since ζ_0 has support in $U \subset P_n$, we have $f_*(\zeta)=\zeta$, which implies that 1 is an eigen **value** of f_* which contradicts a (or, equivalently, b).

Now $b \Rightarrow c$ follows immediately from

Lemma. If the non-periodic points of f are dense in M, then

$$\mu \in \sigma(L),\ \lambda \in \mathbf{C},\ |\lambda|=1 \Rightarrow \lambda\mu \in \sigma(L).$$

Proof. For $r>0$, let $S_r=\{\lambda \in \mathbf{C} : |\lambda|=r\}$.

The lemma asserts that either $\sigma(L) \supseteq S_r$ or $\sigma(L) \cap S_r=\emptyset$. Suppose the contrary. Then we may choose $\mu \in S_r$ which is a boundary point of $\sigma(L)$. Our first step is to show that there does not exist $\varepsilon>0$ such that

$$\|(L-\mu I)\zeta\| \geqq \varepsilon\|\zeta\| \text{ for all } \zeta \in X.$$

For suppose that such an ε exists. Then $(L-\mu I)[X]$ is a proper closed subspace of X, so there exists $x \in X$ whose distance δ from $(L-\mu I)[X]$ is >0. Let $B=\{y \in X : \|y\| \leqq 2\varepsilon^{-1}\|x\|\}$. It is easily seen that if $|\alpha|<\varepsilon/2$, then $x \notin (L-(\mu+\alpha)I)[X \backslash B]$ and if $|\alpha|<\delta\varepsilon/2\|x\|$, then $x \notin (L-(\mu+\alpha)I)[B]$. Thus

$$|\alpha| < \min(\varepsilon/2,\ \delta\varepsilon/2\|x\|)$$

implies x is not in the image of $L-(\mu+\alpha)I$, and hence that $\alpha+\mu \in \sigma(L)$. This contradicts the hypothesis that μ is a boundary point of $\sigma(L)$.

Hence for every $\varepsilon>0$, there exists $\zeta \in X$ such that $\|\zeta\|=1$ and $\|L\zeta-\mu\zeta\| \leqq \varepsilon$. We will complete the proof of the lemma by showing that if $\lambda \in \mathbf{C}$, $|\lambda|=1$ and $\varepsilon>0$, then there exists $\zeta_1 \in X$ such that $\|\zeta_1\|=1$ and $\|L\zeta_1-\mu\lambda\zeta_1\| \leqq \varepsilon$. By what we have just shown, we may choose $\zeta \in X$ such that $\|\zeta\|=1$ and $\|L\zeta-\mu\zeta\| \leqq \varepsilon/4$. Since the non-periodic points of f are dense in M, we may choose a non-periodic point x such that $\|\zeta(x)\| \geqq 1/2$. Let n be a positive integer such that $1/n \leqq \varepsilon/4|\mu|$.

Choose a neighborhood U of x such that $f^k[U] \cap f^l[U]=\emptyset$ for $-n \leqq k < l \leqq n$. Let $\phi : U \to [0, 1]$ be a continuous function with compact support such that $\phi(x)=1$. Define $\psi: M \to [0, 1]$ by

$$\psi = 0 \text{ outside } \bigcup_{-n \leqq k \leqq n} f^k[U]$$

$$= \left(1 - \frac{|k|}{n}\right) \lambda^{-k}(\phi \circ f^{-k}) \text{ on } f^k|U|, \quad -n \leqq k \leqq n.$$

Clearly ψ is continuous. Let $\eta = \psi\zeta \in X = \Gamma(TM \otimes_{\mathbf{R}} \mathbf{C})$. Then

$$\|\eta\| \geqq \|\eta(x)\| = |\psi(x)| \; \|\zeta(x)\| \geqq \tfrac{1}{2}$$

and

$$\begin{aligned} \|L\eta - \mu\lambda\eta\| &= \|(\psi \circ f^{-1})L\zeta - \mu\lambda\psi\zeta\| = \\ &= \|(\psi \circ f^{-1})(L\zeta - \mu\zeta) + \mu(\psi \circ f^{-1})\zeta - \mu\lambda\psi\zeta\| \\ &\leqq \|L\zeta - \mu\zeta\| + |\mu| \; \|\psi \circ f^{-1} - \lambda\psi\| \\ &\leqq \|L\zeta - \mu\zeta\| + |\mu|/n \leqq \varepsilon/2. \end{aligned}$$

Setting $\zeta_1 = \eta/\|\eta\|$, we see from the two inequalities above that $\|L\zeta_1 - \mu\lambda\zeta_1\| \leqq \varepsilon$, as required. This completes the proof of the lemma.

Now we complete the proof of the theorem by showing that $c \Rightarrow d$. Let $\sigma(0) = \{\lambda \in \sigma(L) : |\lambda| < 1\}$ and $\sigma(\infty) = \{\lambda \in \sigma(L) : |\lambda| > 1\}$. By "general spectral theory" ($|1|$, VII), there exists a direct sum decomposition $X = X_{\sigma(0)} \oplus X_{\sigma(\infty)}$ into closed subspaces, which are preserved by L. This decomposition has the following property. Let $L_{\sigma(0)} = L|X_{\sigma(0)}$ (regarded as a mapping $X_{\sigma(0)} \to X_{\sigma(0)}$) and $L_{\sigma(\infty)} = L|X_{\sigma(\infty)}$. Then $\sigma(L_{\sigma(0)}) = \sigma(0)$ and $\sigma(L_{\sigma(\infty)}) = \sigma(\infty)$. (See [1], VII, 3.20 and VII, 3.21). Let $r_0 = \sup\{|\lambda| : \lambda \in \sigma(0)\}$ and $r_\infty = \sup\{|\lambda^{-1}| : \lambda \in \sigma(\infty)\}$. Then $0 < r_0$, $r_\infty < 1$. By [1], VII, 3.4,

$$\lim_{n \to \infty} \sqrt[n]{\|L^n_{\sigma(0)}\|} = r_0 \tag{1}$$

$$\lim_{n \to \infty} \sqrt[n]{\|L^{-n}_{\sigma(\infty)}\|} = r_\infty. \tag{2}$$

From these two equations and the fact that $L = L_{\sigma(0)} \oplus L_{\sigma(\infty)}$ it follows that

$$X_{\sigma(0)} = \{\zeta \in X : \limsup_{n \to \infty} \sqrt[n]{\|L^n(\zeta)\|} \leqq r_0\}. \tag{3}$$

Since L commutes with conjugation, it follows from (3) that $X_{\sigma(0)}$ is preserved by conjugation. Hence there exists a closed subspace Γ_0 of $\Gamma(TM)$ such that $X_{\sigma(0)} = \Gamma_0 \otimes_{\mathbf{R}} \mathbf{C}$.

Similarly there exists a closed subspace Γ_∞ of $\Gamma(TM)$ such that $X_{\sigma(\infty)} = \Gamma_\infty \otimes_{\mathbf{R}} \mathbf{C}$. Clearly $\Gamma(TM) = \Gamma_0 \oplus \Gamma_\infty$. By (3),

$$\Gamma_0 = \{\zeta \in \Gamma(TM) : \limsup_{n \to \infty} \sqrt[n]{\|L^n(\zeta)\|} \leqq r_0\}. \tag{4}$$

From (4), it follows immediately that Γ_0 is a $C(M)$ submodule of $\Gamma(TM)$, where $C(M)$ denotes the ring of continuous real valued functions on M. Similarly one may show that Γ_∞ is a $C(M)$ submodule of $\Gamma(TM)$.

By [3], $\Gamma(TM)$ is a projective finitely generated $C(M)$ module. Since $\Gamma(TM) = \Gamma_0 \oplus \Gamma_\infty$, Γ_0 and Γ_∞ are projective finitely generated $C(M)$ modules. By [3], there exist vector sub-bundles E_- and E_+ of TM such that $\Gamma_0 = \Gamma(E_-)$ and $\Gamma_\infty = \Gamma(E_+)$, and $TM = E_- \oplus E_+$. Since $X_{\sigma(0)}$ and $X_{\sigma(\infty)}$ are preserved by L, Γ_0 and Γ_∞ are preserved by f_*. Hence E_- and E_+ are preserved by Tf.

Take λ with $\sup\{r_0, r_\infty\} < \lambda < 1$. By (1) and (2), there exists $C > 0$ such that $\|L^k_{\sigma(0)}\| \leqq C\lambda^k$ and $\|L^{-k}_{\sigma(\infty)}\| \leqq C\lambda^k$ for all positive integers k. Then for all $v \in E_-$, $\|(Tf)^k(v)\| \leqq C\lambda^k \|v\|$ and for all $v \in E_+$, $\|Tf^k(v)\| \geqq C^{-1}\lambda^{-k}\|v\|$, completing the proof.

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